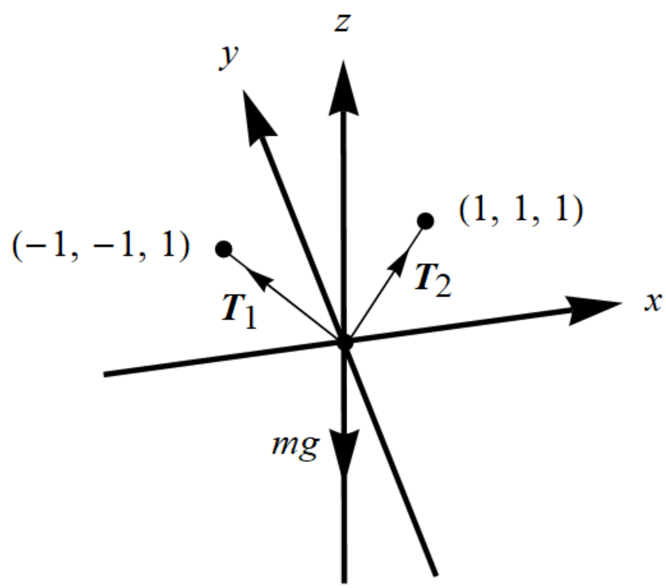
Exercise 35

A 1-kilogram (1-kg) mass located at the origin is suspended by ropes attached to the two points $(1,1,1)$ and $(-1,-1,1)$. If the force of gravity is pointing in the direction of the vector $-\mathbf{k}$, what is the vector describing the force along each rope? [HINT: Use the symmetry of the problem. A 1-kg mass weighs 9.8 newtons (N).]

Solution

Let the tension in each rope be $\mathbf{T}_1$ and $\mathbf{T}_2$ as shown below.



Each one can be written as its respective magnitude multiplied by a unit vector in the rope's direction.

$$\mathbf{T}_1 = \|\mathbf{T}_1\|\hat{\mathbf{u}} = \|\mathbf{T}_1\|\frac{(-1-0)\hat{\mathbf{x}} + (-1-0)\hat{\mathbf{y}} + (1-0)\hat{\mathbf{z}}}{\sqrt{(-1-0)^2 + (-1-0)^2 + (1-0)^2}} = \frac{\|\mathbf{T}_1\|}{\sqrt{3}}(-1,-1,1)$$

$$\mathbf{T}_2 = \|\mathbf{T}_2\|\hat{\mathbf{v}} = \|\mathbf{T}_2\|\frac{(1-0)\hat{\mathbf{x}} + (1-0)\hat{\mathbf{y}} + (1-0)\hat{\mathbf{z}}}{\sqrt{(1-0)^2 + (1-0)^2 + (1-0)^2}} = \frac{\|\mathbf{T}_2\|}{\sqrt{3}}(1,1,1)$$

Apply Newton's second law to the mass.

$$\sum \mathbf{F} = m\mathbf{a}$$

The right side is zero because it's in static equilibrium.

$$\sum \mathbf{F} = \mathbf{0}$$

Substitute the forces on the left side. Here $\mathbf{F}_g$ is the force due to gravity.

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{F}_g = \mathbf{0}$$

$$\frac{\|\mathbf{T}_1\|}{\sqrt{3}}(-1,-1,1) + \frac{\|\mathbf{T}_2\|}{\sqrt{3}}(1,1,1) + (0,0,-mg) = \mathbf{0}$$

Combine the vectors on the left side.

$$\left(-\frac{\|\mathbf{T}_1\|}{\sqrt{3}}, -\frac{\|\mathbf{T}_1\|}{\sqrt{3}}, \frac{\|\mathbf{T}_1\|}{\sqrt{3}}\right) + \left(\frac{\|\mathbf{T}_2\|}{\sqrt{3}}, \frac{\|\mathbf{T}_2\|}{\sqrt{3}}, \frac{\|\mathbf{T}_2\|}{\sqrt{3}}\right) + (0, 0, -mg) = (0, 0, 0)$$

$$\left(-\frac{\|\mathbf{T}_1\|}{\sqrt{3}} + \frac{\|\mathbf{T}_2\|}{\sqrt{3}}, -\frac{\|\mathbf{T}_1\|}{\sqrt{3}} + \frac{\|\mathbf{T}_2\|}{\sqrt{3}}, \frac{\|\mathbf{T}_1\|}{\sqrt{3}} + \frac{\|\mathbf{T}_2\|}{\sqrt{3}} - mg\right) = (0, 0, 0)$$

Match the components on both sides to get a system of equations.

$$\begin{aligned} -\frac{\|\mathbf{T}_1\|}{\sqrt{3}} + \frac{\|\mathbf{T}_2\|}{\sqrt{3}} &= 0 \\ -\frac{\|\mathbf{T}_1\|}{\sqrt{3}} + \frac{\|\mathbf{T}_2\|}{\sqrt{3}} &= 0 \\ \frac{\|\mathbf{T}_1\|}{\sqrt{3}} + \frac{\|\mathbf{T}_2\|}{\sqrt{3}} - mg &= 0 \end{aligned}$$

Solving it yields

$$\|\mathbf{T}_1\| = \frac{\sqrt{3}}{2}mg \quad \text{and} \quad \|\mathbf{T}_2\| = \frac{\sqrt{3}}{2}mg.$$

Therefore,

$$\begin{aligned} \mathbf{T}_1 &= \frac{\|\mathbf{T}_1\|}{\sqrt{3}}(-1, -1, 1) = \frac{mg}{2}(-1, -1, 1) \\ \mathbf{T}_2 &= \frac{\|\mathbf{T}_2\|}{\sqrt{3}}(1, 1, 1) = \frac{mg}{2}(1, 1, 1). \end{aligned}$$